Math 241 **Name:** ______________________
Fall 2019
Midterm 1
9/26/2019
Time Limit: 80 Minutes ID ______________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ______________________

This exam contains 10 pages (including this cover page) and 6 questions.
Total of points is 120.

- Check your exam to make sure all 10 pages are present.
- You may use writing implements on both sides of a sheet of 5"x7" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total:	120	

Boundary value problems:

$$\phi''(x) = -\lambda\phi(x)$$

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}a_n\cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^L f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,dx$

Orthogonality

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0\end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \\ L, & n=m=0\end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0\end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0\end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

1. (20 points) Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

for $0 \leq x \leq 1$ and $t \geq 0$ subject to boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = 0$$

and initial condition $u(x,0) = \cos(\frac{\pi}{2}x) + 4\cos(\frac{5\pi}{2}x)$.

2. (20 points) Consider the 1D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x.$$

with $0 \leq x \leq 1$ and $t \geq 0$ subject to boundary conditions

$$u_x(0,t) = 1, \quad u_x(1,t) = \beta$$

and initial condition $u(x,0) = f(x)$.

1. For what value of β is there an equilibrium solution?
2. Determine the equilibrium solution.

3. (20 points) Solve the Laplace equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the disk $D = \{(x,y)|x^2+y^2 \leq 4\}$ subject to boundary condition

$$\frac{\partial u}{\partial r}(2,\theta) = 32\cos(4\theta) - 8\sin(2\theta).$$

4. (20 points) Solve Laplace equation inside a rectangle $0 \le x \le L, 0 \le y \le H$ with boundary conditions

$$u(0,y) = 0, \quad u(L,y) = g(y), \quad u(x,0) = 0, \quad u(x,H) = 0.$$

5. (20 points) Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + tx.$$

with $0 \le x \le 1$ and $t \ge 0$ subject to boundary conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = 0$$

and initial condition $u(x,0) = \frac{1}{2}x^2 - \frac{1}{3}x^3$. Define the heat energy by

$$E(t) = \int_0^1 u(x,t)\,dx.$$

Find $E(t)$.

6. (20 points) The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

describes the temperature distribution $u(x,t)$ of a 1D rod $0 \leq x \leq \pi$ suffering some heat loss. Find a solution to the PDE with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0$$

and initial condition $u(x,0) = 5 + \cos x + \cos 3x$.

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.